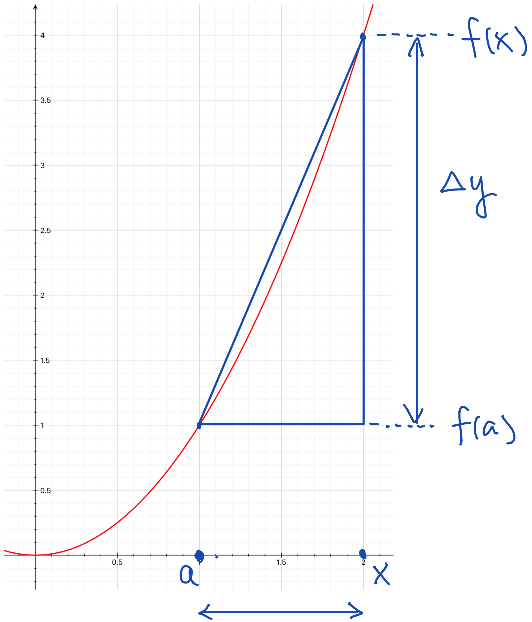
Math 1510 Week 4

## Rate of change / Slope

Goal: Find slope of the graph

$$y = f(x) = x^2 \text{ at } a = 1$$




$\Delta x$ = Difference in $x$ = $h$

## Approximation:

$$\text{Slope of } f(x) \text{ at } x=1 \approx \frac{f(2)-f(1)}{2-1} = 3$$

$$\approx \frac{f(1.1)-f(1)}{1.1-1} = 2.1$$

Better Approximation
$\because$ 1.01 and 0.99 are closer to 1

$$\left\{ \begin{array}{l} \approx \dfrac{f(1.01)-f(1)}{1.01-1} \approx 2.01 \\[2ex] \approx \dfrac{f(0.99)-f(1)}{0.99-1} \approx 1.99 \end{array} \right.$$

## Rmk

① Slope of the line joining $(a, f(a))$ and $(x, f(x))$

$$= \frac{f(x)-f(a)}{x-a} = \frac{f(a+h)-f(a)}{h} \quad \text{where } h = x-a$$

② As $x \to a$, slope $\to 2$

$\therefore$ Slope of $f(x)$ at $a$ should be 2.

# Differentiability

**Defn** (First Principle)

$f(x)$ is called differentiable at $a$ if

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$f'(a)$ is called the derivative of $f$ at $a$.

① One can also define

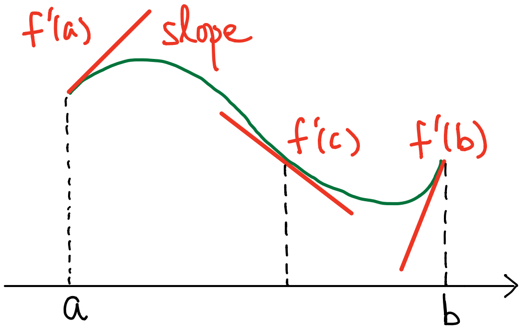
Left-hand derivative $\quad Lf'(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$

Right-hand derivative $\quad Rf'(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$

| | | |
|---|---|---|
| $f(x)$ is differentiable at $a$ | $\Longleftrightarrow$ | $Lf'(a)$ and $Rf'(a)$ exist and equal |

If so, $f'(a) = Lf'(a) = Rf'(a)$

② If $D_f = [a, b]$, then $f$ is said to be differentiable at $\begin{cases} a & \text{if } Rf'(a) \text{ exists} \\ b & \text{if } Lf'(b) \text{ exists} \end{cases}$




③ The derivative can be viewed as a function $f'(x)$ by varying $a$ in $f'(a)$

④ Other notations: If $y = f(x)$

$$f'(x) = \frac{df}{dx} = \frac{dy}{dx} = y'$$

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a} = y'\big|_{x=a}$$

eg 1 Let $f(x) = |x|$.

Find $f'(-2)$ and $f'(0)$ from definition.

Sol

For $f'(-2)$,

$$f'(-2) = \lim_{h\to 0} \frac{f(-2+h)-f(-2)}{h}$$

$$= \lim_{h\to 0} \frac{|-2+h|-|-2|}{h}$$

$$= \lim_{h\to 0} \frac{-(-2+h)-2}{h}$$

$$= \lim_{h\to 0} \frac{-h}{h}$$

$$= -1$$

When $h \approx 0$

$-2+h < 0$

$\Rightarrow |-2+h| = -(-2+h)$

For $f'(0)$,

$Lf'(0)$

$$= \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^-} \frac{|h|-|0|}{h}$$

$$= \lim_{h\to 0^-} \frac{-h-0}{h} \quad (\because h<0)$$

$$= -1$$

$Rf'(0)$

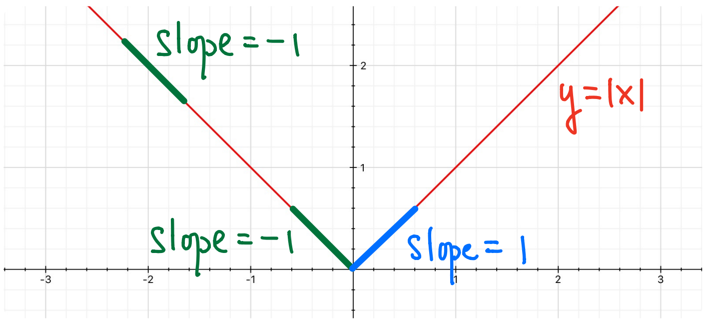
$$= \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{|h|-|0|}{h}$$

$$= \lim_{h\to 0^+} \frac{h-0}{h} \quad (\because h>0)$$

$$= 1$$

$Lf'(0) \neq Rf'(0) \Rightarrow f$ is not differentiable at 0.



eg 2 Let $g(x) = 2x^3 - 3$.

Find $g'(1)$ from definition

Sol

$$g'(1) = \lim_{h \to 0} \frac{g(1+h) - g(1)}{h}$$

$$= \lim_{h \to 0} \frac{2(1+h)^3 - 3 - [2(1)^3 - 3]}{h}$$

$$= \lim_{h \to 0} \frac{2(1 + 3h + 3h^2 + h^3) - 2}{h}$$

$$= \lim_{h \to 0} 6 + 6h + 2h^2$$

$$= 6 + 6(0) + 2(0)^2$$

$$= 6$$

eg 3 Let $f(x) = \frac{1}{\sqrt{x}}$, $x > 0$. Find $f'(x)$ from definition

Sol

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}$$

$$= \frac{-1}{\sqrt{x}\sqrt{x+0}(\sqrt{x} + \sqrt{x+0})}$$

$$= -\frac{1}{2x^{\frac{3}{2}}}$$

# Derivatives of some basic functions (Let a,c be real constants)

Exercise

Prove them!

## Constant functions

$$\frac{d}{dx}(c) = 0$$

## Power functions

$$\frac{d}{dx}(x^a) = ax^{a-1}$$

## Exponential functions

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = (\ln a)\, a^x \quad (a > 0)$$

## Logarithm functions

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \quad (\ln x = \log_e x)$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \quad (a > 0)$$

## Trigonometric functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

## Inverse Trigonometric functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

# Some Rules of differentiation

If $f, g$ are differentiable at $a$, then $f \pm g$, $fg$ and $\frac{f}{g}$ (if $g(a) \neq 0$) are differentiable at $a$ too. Also,

① $(f \pm g)'(a) = f'(a) \pm g'(a)$

② $(cf)'(a) = cf'(a)$ for a constant $c$.

## Product Rule

③ $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

## Quotient Rule

④ $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$

eg $\frac{d}{dx}\left(4x^3 - \frac{6}{\sqrt{x}} + \cos x\right)$

$$= \frac{d}{dx}(4x^3) - \frac{d}{dx}\left(\frac{6}{\sqrt{x}}\right) + \frac{d}{dx}(\cos x)$$

$$= 4\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x^{-\frac{1}{2}}) + \frac{d}{dx}\cos x$$

$$= 4(3x^2) - 6\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) - \sin x$$

$$= 12x^2 + 3x^{-\frac{3}{2}} - \sin x$$

eg $[2^x(x^2 + x + e^2)]'$

$$= (2^x)'(x^2 + x + e^2) + 2^x(x^2 + x + e^2)'$$

$$= (\ln 2)2^x(x^2 + x + e^2) + 2^x(2x + 1 + 0)$$

Rmk All steps above can be skipped.

eg

$$\left(\frac{x\ln x}{\sin x}\right)'$$

$$= \frac{(x\ln x)'\sin x - (x\ln x)(\sin x)'}{\sin^2 x}$$

$$= \frac{\left[(1)\ln x + x\cdot\frac{1}{x}\right]\sin x - (x\ln x)\cos x}{\sin^2 x}$$

$$= \frac{(\ln x + 1)\sin x - (x\ln x)\cos x}{\sin^2 x}$$

Pf for $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Recall: $e = \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}$

$$\frac{d}{dx}(\ln x) = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\frac{h}{x}\ln\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$$

$$= \frac{1}{x}\ln\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right]$$

$$= \frac{1}{x}\ln e$$

$$= \frac{1}{x}$$

## Pf for $\frac{d}{dx}(\sin x) = \cos x$

Recall: $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{2\cos(x+\frac{h}{2})\sin\frac{h}{2}}{h}$$

$$= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\cdot\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= \cos(x+\tfrac{0}{2})\cdot(1)$$

$$= \cos x$$

Try to prove $\frac{d}{dx}(\csc x) = -\csc x\cot x$

using the same formula above

## Pf of $(f+g)' = f' + g'$

$$(f+g)'(x)$$

$$= \lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= f'(x) + g'(x)$$

Pf of Product Rule: $(fg)' = f'g + fg'$

$$(fg)'(x) = \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h}$$

$$= f'(x)g(x) + f(x)g'(x)$$

Q Why $\lim_{h\to 0} g(x+h) = g(x)$? Is $g$ continuous?

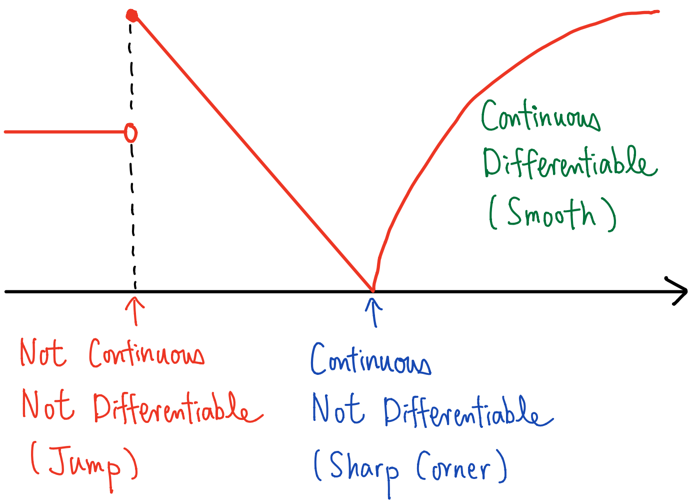
A Yes! $g$ is differentiable $\Rightarrow$ $g$ is continuous (see next page)

**Thm** If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

Differentiable $\Rightarrow$ Continuous

Equivalently,

NOT Continuous $\Rightarrow$ NOT Differentiable




Pf

Suppose $f$ is differentiable at $a$. Then

$$\lim_{x\to a} f(x)$$

$$= \lim_{x\to a} f(x) - f(a) + f(a)$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{x-a}\cdot(x-a) + f(a)$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{x-a}\cdot \lim_{x\to a}(x-a) + f(a)$$

$$= f'(a)(a-a) + f(a)$$

$$= f(a)$$

$\therefore$ $f$ is continuous at $a$